"The 'Bombardon' Floating Breakwater." *

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Introduction.

Large scale planning for the invasion of Northern France was commenced in 1942. The artificial-harbour element in that planning arose out of the lessons learned from the Dieppe raid. The practical impossibility of capturing a working port and the tremendous risks involved in the alternative of maintaining supply lines across open beaches had created the demand for artificial harbours. In March 1943, the Combined Chiefs of Staff in a memorandum addressed to the First Sea Lord stated, "this project (artificial harbours) is so vital to 'Overlord' (the invasion operation) that it might be described as the crux of the whole operation." In April and May 1943, a possible solution of the problem appeared in the form of the floating breakwater. Arising out of the Quebec Conference of 1943 it was decided to construct the "Mulberry" harbours from a combination of blockships, "Phoenix" units, and floating breakwaters. In 6 months over a mile of floating breakwater was designed, assembled, and successfully tested off the Dorset coast. Over 2 miles of floating breakwater formed an integral part of the original harbour at Arromanche and Saint Laurent. They met all the staff requirements and, in combination with the blockships and while the Phoenix breakwaters and "spud" piers were being assembled, provided invaluable shelter and enabled the necessary build-up to be achieved on shore during the first critical fortnight.

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† Mr Lochner held the rank of Lieutenant-Commander at the time the work described in this Paper was carried out.

Theory.

Knowledge of marine waves has made great strides since the publication of Dr. Vaughan Cornish's classic work.[1] Largely owing to the researches of Airey, Stokes, Suthon, and other investigators, a complete and accurate theory of marine waves now exists. It is now possible to forecast the height, length, and period of sea and swell which may be generated by a given wind-strength, and to make accurate predictions of the maximum size and length of wave which can be generated in any given locality. It is possible from considerations of the area and depth of water to arrive at a very close estimate of the most severe conditions to be experienced by harbour works in any given locality due to wave action alone.

The existence of this fund of accurate knowledge was the first essential in the successful production of the Bombardon floating breakwater. The operation of the breakwater depends upon correctly combining four well-known principles, namely :

(1) that the maximum height, length, and period of the waves in any given locality are determined by the geographical configuration of that locality ;
(2) that the waves of the sea are relatively skin deep ;
(3) that the amplitude of oscillation in an oscillatory system having a long natural periodicity is small when subjected to a forced oscillation of relatively short periodicity ; and
(4) that a floating object may, under suitable circumstances, be designed to have long natural periods in each of its three modes of oscillation.

These four principles will now be discussed in greater detail.

Waves.

Marine or gravitational waves were investigated mathematically by Airey, who developed a theory based upon the assumption that the motion of the particles in a system of uniform travelling waves was wholly circular or elliptical and non-translatory. From that theory Airey deduced the following mathematical expressions for the co-ordinates of a particle of the fluid acted upon by a system of uniform travelling waves moving from

$$x = +\infty \text{ to } x = -\infty$$

$$X = a\frac{\cosh K(y+H)}{\sinh KH}\cos(Kx - \sigma t)$$

$$Y = a\frac{\sinh K(y+H)}{\sinh KH}\sin(Kx - \sigma t)$$

[1] Vaughan Cornish, " Waves of the Sea and Other Water Waves ". T. Fisher Unwin, London, 1910.

where $K = \frac{2\pi}{\text{wave-length}}$, H denotes the mean depth of water, σ denotes the angular velocity, and a denotes the amplitude (half the wave-height) at the surface of the fluid.

From these equations it is very easy to determine that a particle at a mean depth y below the mean surface will move in an elliptical orbit whose major and minor axes are, respectively,

$$2a\,\frac{\cosh K(y+H)}{\sinh KH} \text{ and } 2a\,\frac{\sinh K(y+H)}{\sinh KH}.$$

Where H is greater than half a wave-length, these expressions reduce to

$$2ae^{Ky}$$

Since y is measured in the downward (negative) direction the value of this factor, which represents the diameter of an orbit at depth y, will be $2a$ at the surface and will diminish rapidly until, at a depth equal to the wave-length, there will be less than two-thousandths of the movement at the surface.

The radius of the orbit of a particle at various depths is shown graphically in curve A of *Fig. 1.*

It is also fairly easy to determine from the above theory that, where the depth of water exceeds half a wave-length, the energy contained in one complete wave of a uniform system of travelling waves is equal to $\frac{1}{2}g\rho a^2\lambda$ per unit length of wave front, where $g = 32{\cdot}16$, ρ denotes the density of the fluid, and λ denotes the wave-length. The energy in the layer of fluid contained between the surface and a depth D below the mean surface is likewise $\frac{1}{2}g\rho\lambda a^2(1 - e^{-2KD})$, whilst the amount of energy remaining between the depth D and the bottom is $\frac{1}{2}g\rho\lambda a^2 e^{-2KD}$. This latter expression also represents the amount of energy passing underneath a barrier which extends to a depth D and not to the bottom. Values for this factor are shown graphically in curve B of *Fig. 1.*

The angular velocity of the particles in such a wave is determined from the equation

$$\sigma = \left(\frac{2\pi g}{\lambda}\right)^{\frac{1}{2}}.$$

From this equation a very simple rule may be deduced for deep water waves which enables the wave-length and period to be related. If the wave-length, measured from crest to crest, is expressed in feet and the period in seconds, then :—

$$\text{wave-length in feet} = 5{\cdot}15 \times (\text{period in seconds})^2.$$

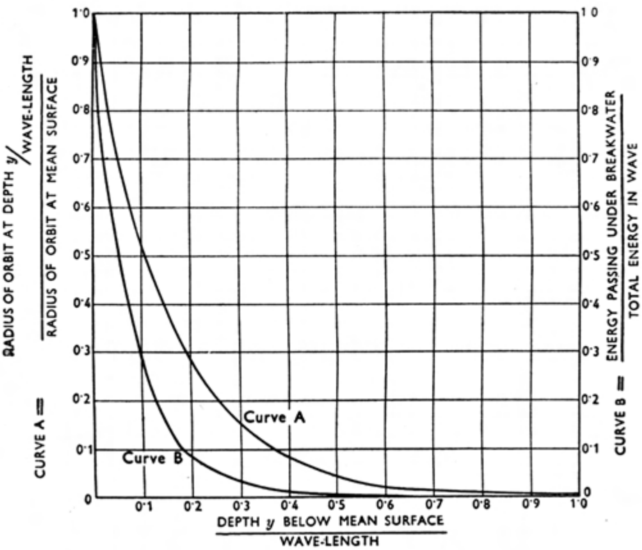
This relation holds in deep water, and, approximately, in water deeper than one-fifth of the wave-length.

Since the above-mentioned theoretical development by Airey, others have examined the problem of gravitational waves, notably Stokes and

Suthon, and it is now possible to determine the height, length, pressure, and period of waves under widely varying conditions.

One of the most interesting results of the later work is to establish with greater accuracy the relation between the strength and duration of the wind, the distance over which it is operative, and the size and period of the waves generated. It is known, for example, that the length of waves is dependent not only upon the velocity of the wind but also upon the area of water affected by its passage. The greatest hurricane that ever blew would fail to raise Atlantic rollers in the North Sea, and similarly, a local wind blowing across a few miles of Atlantic would fail to generate waves longer than those found, say, in the Baltic, even though it blew

Fig. 1.



RELATION BETWEEN DEPTH AND WAVE-MOTION.

at 100 miles per hour or more. In order to generate a wave of a given length, height, period, and contained energy, there must be sufficient sea-room for the wind to impart the necessary energy to the water of the wave. In the case of the longer waves, this requires hundreds and in some cases thousands of miles of unobstructed deep water. As a consequence of this natural law the maximum period of waves in the smaller enclosed waters is limited by the maximum distance over which the wind may blow and not the maximum velocity at which it may blow. In such areas as the southern North Sea, the Baltic, the Mediterranean, the Great Lakes of Canada, and in the case of other enclosed waters this rule applies and a maximum period for each of these areas can be calculated from considerations of the distance between shores and depth of water alone with the full knowledge that, however hard the wind may blow, this period and corresponding wave-length cannot be exceeded.

Nature also sets a limit to the height of sea and, in general, this will not exceed one-fifteenth and in rare cases one-tenth of the wave-length. Beyond a ratio of one-seventh, the mechanics of gravitational waves are such as to cause the wave to break and in breaking to dissipate a substantial part of its energy as heat. Similarly, there is, for every given depth, a maximum possible height of wave beyond which breaking and dissipation of energy must occur. One of the methods of measuring depth of shallow water from the air depends in fact upon this physical law.

The approach, then, to the problem of building harbour works is simplified to-day by the fact that the engineer may, if he desires, arrive at an exact estimate of the characteristics of the seas he may expect to experience, while the designer of a floating harbour will be able accurately to determine the maximum period he must design to meet and the depth to which he must take his breakwater in order to reflect the desired quantity of wave energy.

Oscillatory Systems

Any mechanical system containing elastically connected, freely moving masses, if disturbed and then left free, will oscillate, after an initial transitory interval, with a definite natural periodicity depending upon the values of mass and elasticity alone and not upon the nature or periodicity of the original disturbance. An electrical circuit possessing inductance and capacity will behave in an analogous manner. Even mixed mechanical and electrical oscillatory systems will obey the same general laws.[1]

If an external disturbing force of uniform periodicity is applied to such a mechanical oscillatory system, the behaviour of the elements of the system will depend largely upon the relation between the natural periodicity of the system and the periodicity of the external disturbing force. When the external period is much longer than the natural period, the masses will tend to move with almost the same amplitude and phase as the external force. When the external period is much shorter than the natural period, the masses will tend to remain stationary and any movement which then takes place will be out of phase with the external force. When the two periodicities are equal the condition of resonance occurs and the movements of the masses will be greater and may be much greater than the amplitude of the disturbing force and will be limited solely by the frictional damping present in the system.

[1] R. A. Lochner, "Torsional Vibration of Shafts and Shaft Systems". J. Instn Elec. Engrs, December 1926.

These relations are expressed in the well-known equation for a system having a mass m, a damping coefficient Q, an elasticity coefficient R, a natural periodicity P_N, and a disturbing force of amplitude a and periodicity P_E

$$m\frac{d^2s}{dt^2} + Q\frac{ds}{dt} + Rs = a\cos\frac{2\pi}{P_E}t.$$

The solution of this equation may be written in the form

$$b\cos\left(\frac{2\pi}{P_E}t - \epsilon\right)$$

where $$b = \frac{a}{R\sqrt{\left(1-\frac{P_N^2}{P_E^2}\right)^2 + \frac{Q^2}{mR}\cdot\frac{P_N^2}{P_E^2}}}$$

and $$\tan\epsilon = \frac{2\pi Q}{R(P_E^2 - P_N^2)}.$$

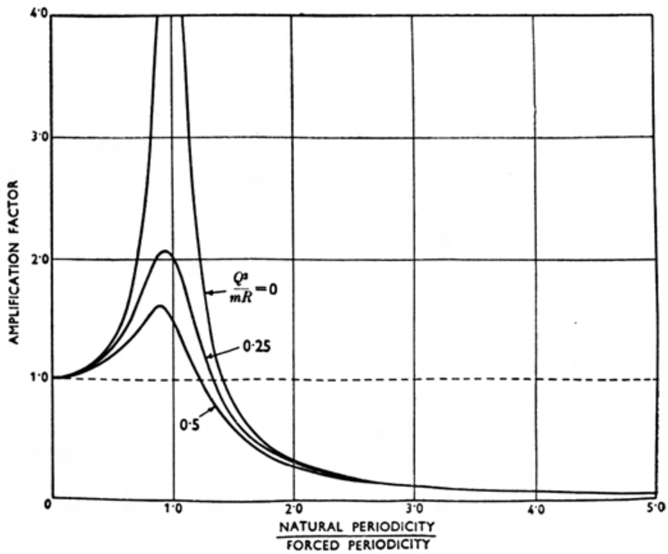
The amplitude of movement of the mass is equal, therefore, to the amplitude of the disturbing force multiplied by the factor:

$$\frac{1}{\sqrt{\left\{1-\left(\frac{P_N}{P_E}\right)^2\right\}^2 + \frac{Q}{mR}\left(\frac{P_N}{P_E}\right)^2}}$$

where $P_N = 2\pi\sqrt{\frac{m}{R}}$ and denotes the natural period of the oscillatory system. By making m large and R small, and increasing Q as much as possible, it is obvious that the above-mentioned amplification factor can be made considerably less than unity, and the amplitude of movement of the mass may be reduced to a small percentage of the amplitude of the disturbing force. The value of this amplification factor for various ratios of $\frac{P_N}{P_E}$ and for $\frac{Q^2}{mR} = 0$, 1, and 2, is shown in the three curves in *Fig. 2.*

If the external disturbing force is a train of gravitational waves and the mass m is a breakwater wall, then it is obvious that if m can be prevented from moving, the train of waves on reaching the wall will suffer total reflexion and any water on the lee side of the wall will be unaffected by the passage and reflexion of the wave train. This effect can be produced by fixing the wall to the surface of the earth so that it virtually possesses infinite mass relative to the waves. Of this form is the ordinary stone or reinforced-concrete wall. But a great deal of the material in such a wall, from the point of view of reflecting wave energy, is wasted. As mentioned in the previous section, the energy of gravitational waves is mostly concentrated in the surface layer, and a reflecting wall, in order to be effective, need only descend to a depth equal to about 15 to 20 per cent. of the wave-length. The difficulty with floating walls has been to keep them stationary and make them act as reflectors. By utilizing the principle briefly described in this section, and giving to such a floating wall those values of m, Q, and R which reduce the above-mentioned amplification factor to a small fraction of unity, it is possible to make such a floating wall remain relatively stationary and operate as a wave reflector. The primary condition, as an examination of the equation for the amplification factor will show, is that the natural period of oscillation of the floating structure shall be considerably longer than the maximum periodicity of

Fig. 2.



Amplification Factor.

the longest wave which the floating breakwater has to reflect and against which it must provide protection.

Long-Period Floating Structures.

Floating structures are usually considered as being capable of three modes of oscillation corresponding to the motions of rolling, pitching, and heaving. A floating structure which is to reflect wave energy must have the requisite long natural periodicities in each of these three modes of oscillation. Hitherto, this condition has only been possible in the conventional design of ships hull, by using a very large mass of material compared with the mass of the wave suppressed. Thus to give protection against waves of 100 feet length would require a conventional hull section corresponding to a ship of over 10,000 tons displacement. Apart from the almost insuperable difficulties of mooring such a design of floating breakwater, its capital cost would be prohibitive. It is possible, however, by suitable design to obtain the required long natural periods with greatly reduced expenditure of material, and when this is done the cost of an effective floating breakwater is reduced, in the normal case, to a figure much below that for the fixed type.

To obtain a long natural period, it is necessary to combine large mass with small elasticity. In a floating structure the elasticity is represented by the increase or decrease of buoyancy accompanying any of the three modes of oscillation. For example, if the floating structure is immersed below its normal flotation mark by a uniform amount along its length (corresponding to the motion of heave), there will be an increased upward thrust or restoring force due to the increased immersion. If released, the floating structure will rise and its mass will carry it beyond its normal flotation marks until the excess of weight over displacement decellerates the mass. In this manner a buoyant floating structure behaves in the same way as a weight suspended by a spiral spring, the elasticity of the spring being replaced by the restoring force represented by the balance between weight and displacement. It follows that to obtain a long period it is necessary to increase the mass and simultaneously to reduce this restoring force, but in the conventional design of hull these are conflicting requirements. To increase the mass involves increase of weight and, unless the draught is increased, this involves, in the normal hull, an increase of the restoring force, by reason of the increase of beam and displacement to compensate for the increase of weight. In consequence hull dimensions have to assume very large proportions before periods are reached sufficient to ensure reflexion of the longer waves. The same difficulties apply substantially in normal hull design to the periodicities of roll and pitch.

In the floating breakwater, these difficulties have been surmounted either by using the water, in which the breakwater floats, to supply the necessary mass, or by reducing the restoring force to very small proportions by employing flexible sides, or by a combination of these factors. In the case of the type in which the water supplies the mass, the restoring force and displacement are then only proportional to the weight of the enclosing structure. By these means very long periods can be obtained and waves reflected with less than one-thirtieth of the expenditure of material required for the same purpose in a conventional hull design.

Development of the Bombardon Breakwater.

The first model of a floating breakwater tested in May 1943 was one built in accordance with the above principles and equipped with flexible sides. This type is interesting for the present purpose only in so far as it helped to prove the above-mentioned theories and to establish that floating breakwaters could provide calm water as efficiently as fixed breakwaters. Three full-scale flexible-sided breakwaters were launched and tested in October and November of 1943. Each was 200 feet long with 12 feet beam and 16½ feet draught. The hull consisted of four rubberized canvas envelopes placed one inside the other and enclosing three air compartments, each running the full length of the hull. The envelopes were attached to and supported a 700-ton solid reinforced-concrete keel. The air pressure in the three compartments was adjusted to coincide approximately with the mean hydrostatic pressure on the outside of the respective envelopes. In that way a form of hull side was obtained which moved in or out under any temporary unbalance between those two pressures corresponding to any alteration of immersion depth. In consequence, the restoring force with that type of hull was only a small fraction of that for a rigid-sided hull of the same displacement and the periodicities were correspondingly lengthened.

That earlier prototype was notable in two ways. First, due to the flexible nature of the sides of the breakwater the reflexion of wave energy took place substantially at the anti-node, and secondly, the three units were, to the best of the Authors' knowledge and belief, the largest flexible-sided vessels ever built. The construction of the great envelopes for the Admiralty, by the Dunlop Rubber Company Limited, was a notable and praiseworthy achievement and went far to establish the validity of the general theory of floating breakwaters. One of them is illustrated in *Fig. 3*.

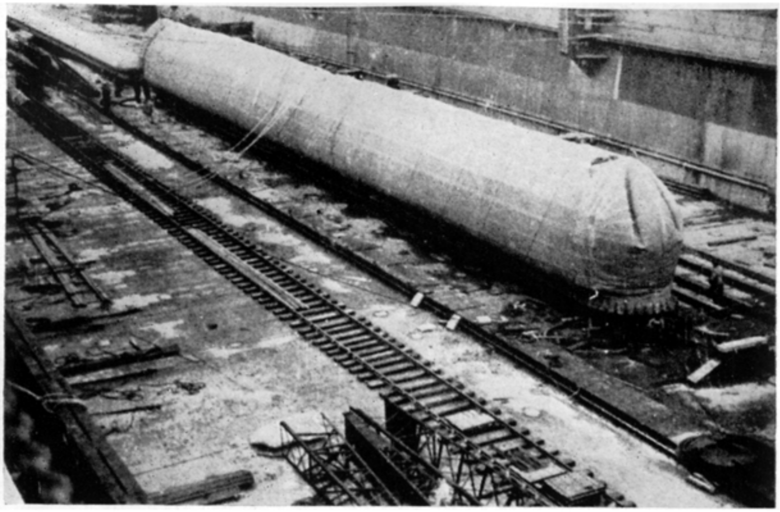
The flexible-sided breakwater was not adopted for operation "Overlord" because of the vulnerability of its fabric sides, and after June 1943 the theoretical and experimental work was mainly devoted to the development of a rigid-sided counterpart. That embodied the second of the two constructional principles enumerated above, namely, the enclosure of a large mass of water within a relatively light enclosing structure in such a way that the restoring force was reduced to a minimum.

The first models of the Bombardon floating breakwater were tested in June 1943, and by the end of August sufficient data had been assembled to establish the correctness of the theories applying to the rigid-sided type. Over three hundred model-tests of the rigid type were made before full-scale designs were put in hand. Those tests were made at the Admiralty Experimental Works at Haslar and were directed to checking the theory of wave suppression by floating breakwaters and to determining the towing and mooring data necessary for the full-scale operation. The one-tenth-scale models on which the full-scale designs were ultimately based are shown in *Figs 4*.

The results of those model-tests agreed very closely with the mathematical theory and later agreed with the full-scale results when they became available.

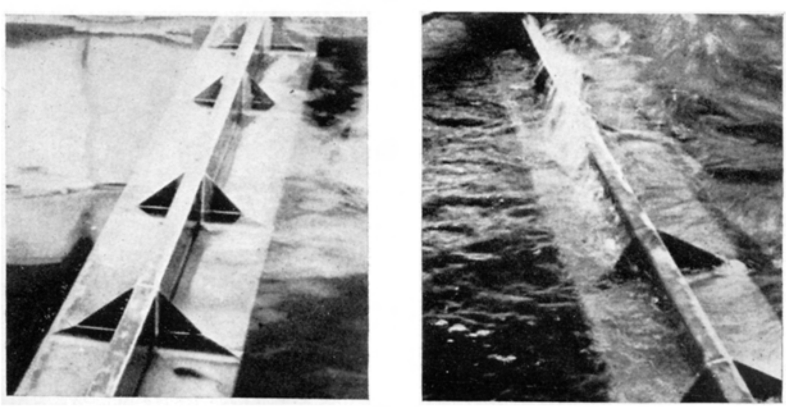
The mathematical theory of floating breakwaters is complex and the

Fig. 3.



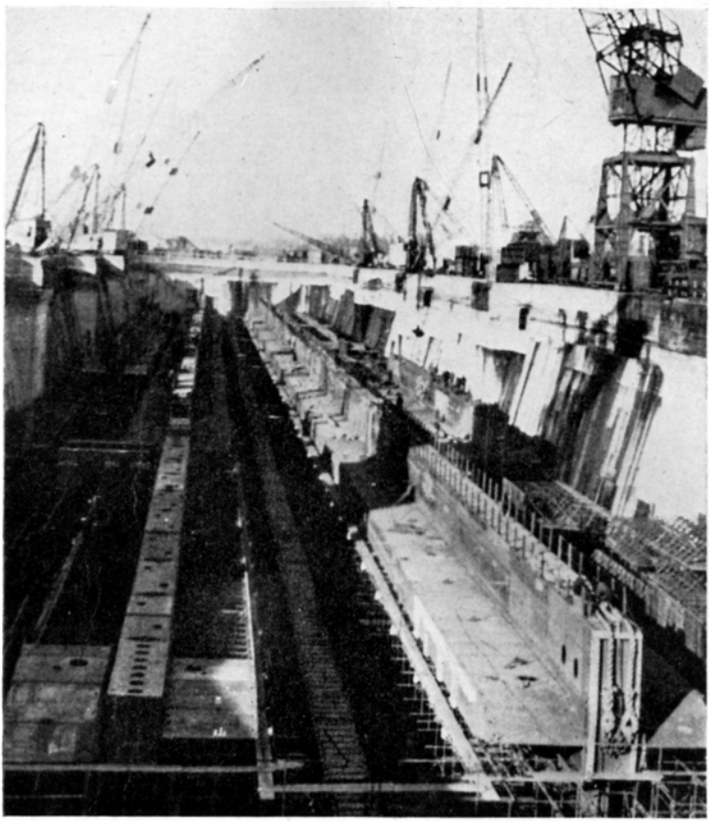
Flexible-Sided Breakwater Unit.

Figs 4.



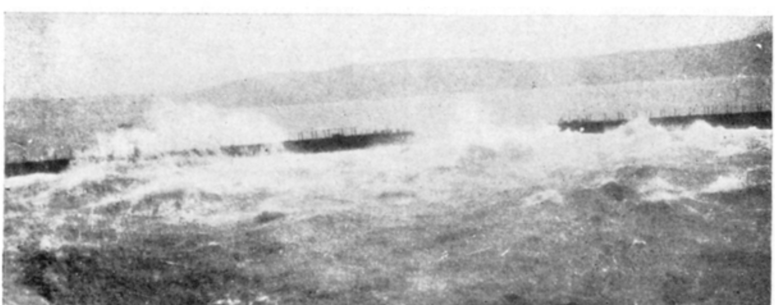
Scale Model of Basis for Ultimate Design.

Fig. 5.



BOMBARDON UNITS UNDER CONSTRUCTION.

Fig. 6.



ON TEST DURING A GALE.

analysis of the model-results was correspondingly complicated. It would be out of place in a short Paper of this nature to develop this theory *in extenso*, but an extract is given in the Appendix. The net result, however, of the extremely concentrated work performed was to show that it was possible to construct a floating breakwater which would suppress waves of the maximum size anticipated in operation "Overlord" for an expenditure of about $1\frac{1}{4}$–$2\frac{1}{2}$ tons of steel per foot of breakwater frontage. That represented an expenditure of less than one-tenth of that required for any other possible method.

The Full-Scale Floating Breakwater.

The decision to proceed with a full-scale floating breakwater as an integral part of the Mulberry harbours was taken at Washington on the 4th September, 1943, and was signalled to England on the same day. The design and construction of the floating breakwaters and the assembly, transport, and siting of the entire Mulberry harbours were to be Admiralty responsibilities.

At the date when that decision was taken there remained little more than 8 months to the original date of D-day. In that period the remainder of the theory mentioned above had still to be formulated, a considerable number of the three hundred model-tests had still to be made, the full-scale designs and production plans had to be prepared and over 4 miles of an entirely new and, as yet, untried form of floating breakwater had to be built, assembled, tested, and then finally sited 100 miles from the English coast and under the fire of the enemy's guns. That it was completed and ready to move off with the rest of the invasion fleet is a remarkable tribute to all who took part in this great enterprise.

The staff requirements laid down by the Combined Chiefs of Staff for the floating breakwater portion of the Mulberry harbours were as follows :—

(1) Sufficiently mobile to be towed across the Channel and provide some sheltered water by D-day + 4.
(2) To be completed in all respects by D-day + 14.
(3) To be strong enough to withstand winds up to, and including, force 6.
(4) To be capable of being moored in water deep enough to provide shelter for fully laden liberty ships.
(5) To be ready in all respects by May 1944.

It will be noticed that the breakwater was to be designed to withstand conditions up to winds of force 6 only. The fact that much stronger winds blow at certain times of the year in the English Channel was fully appreciated at the time that that condition was laid down. But it was clear from all the available statistics that the probability of winds over force 6 in June in the Channel was so low that for practical purposes their occurrence could be ignored. Had any other decision been made, lack of time and materials would have made completion of the project impossible.

The first estimate of the height and length of sea corresponding to force 6 given to the designers of the harbour equipment was 8 feet and 100 feet respectively. Those estimates were made on the preliminary assessment of the physical factors corresponding to the first invasion plan. At a later stage and due to changes in those plans it became necessary to increase the figures to 10 feet and 150 feet respectively. The final production designs for the Bombardon floating breakwater were based on the latter estimate, which was found to agree closely with the actual height and length of sea measured under force 6 wind conditions at the trial and operational sites.

After a survey of the available production resources and the very heavy demands of the other services, the Admiralty decided to endeavour to meet the requirements of the Chiefs of Staff by means of a mass produced pre-fabricated construction of floating breakwater, the components of which would be bolted together in the final assembly. The choice of a bolted construction is one no naval architect would countenance in normal times. Its choice for the operation was one dictated solely by the impossibility of obtaining the requisite amount of riveting or welding labour to enable the more normal forms of construction to be adopted.

The original full-scale design contained approximately 250 tons of steel. The overall dimensions of each unit were 200 feet length, 25 feet 1 inch beam, 25 feet $1\frac{3}{4}$ inches hull depth and 19 feet draft. The cross-section was roughly the form of a Maltese cross. The top half of the vertical arm of the cross was mainly built up of watertight buoyancy compartments made from welded $\frac{1}{4}$-inch mild-steel plate, whilst the bottom half and the two side arms were constructed from mild-steel angles and plate in bolted sections. The bottom and side arms filled with water upon launching and provided the requisite mass. The effective beam at the water-line was less than 5 feet which resulted in a restoring force per foot of change of draft of under 30 tons. That restoring force should be compared with the 1,500 tons of water which the unit contained inside and outside the arms of the cross.

The general nature and appearance of the units may be gathered from *Fig. 5* (facing p. 265), which shows groups of Bombardons under construction in the King George V dock at Southampton.

In order to construct a floating breakwater wall, it was necessary to moor a number of the units in line ahead. Normally, when mooring ships to head and stern moorings, a gap is left between adjacent ships which approximates to the length of the ships themselves. In the case of the floating breakwater such an arrangement would have resulted in half the wave energy passing through the gaps between units and rebuilding inside the harbour to a wave of three-quarters the original height. To reduce that effect it was necessary to work with much smaller gaps between units and after a number of trials and calculations it was decided to use a 50-foot gap. That relatively small gap was successfully achieved by mooring Bombardons in pairs between mooring buoys, the couplings between Bombardons being composed entirely of twin 18-inch cable-laid manilla rope. The couplings absorbed the relative movement between units without shock and enabled the gaps to be successfully maintained. The reduction of the gaps to 20 per cent. of the total breakwater length enabled a corresponding reduction to be made in the energy filtering through between the units. In order to reduce that energy still further, however, it was also decided to use two parallel lines of Bombardons spaced 800 feet apart. That arrangement indicated a theoretical reduction of wave-height to approximately 30 per cent. and a reduction of wave energy to one-tenth of the original incident wave; figures which were almost exactly reproduced in practice.

The problem of mooring a large number of such units in close proximity was solved by the adoption of a system of laying which ensured accurate spacing of the mooring buoys.

The initial lay consisted of a 5-inch flexible ground-wire with 1,000-lb. sinkers at equally spaced intervals to which wire risers and spherical floats were attached. The floats were then replaced by mooring buoys and the seaward and leeward anchor cables were attached. The seaward leg was secured to two 3-ton and one 5-ton mushroom anchors and one 8-ton concrete clump, and the leeward leg to one 3-ton mushroom anchor. Those anchors were chosen mainly to suit the available materials and to keep the individual weights down to a minimum consistent with rapid laying. A somewhat different type of mooring would be used under peace-time conditions. As soon as the moorings were in position, the Bombardons were attached to the buoys in pairs by means of their manilla connectors. By means of that relatively simple lay-out, twenty-six moorings were laid and over 2 miles of floating breakwater were completely assembled off the French coast in 6 days.

The first test of a full-scale floating harbour took place in Weymouth bay at the beginning of April 1944. The harbour consisted of an outer line of nine and an inner line of six units moored in the manner just described. Elaborate arrangements were made for recording both visually and photographically the height, length, and period of the waves on the seaward and leeward sides of the breakwater, and new instruments had to be developed for the purpose.

At the time that those wave experiments were commenced, the instrument used almost exlusively by the Admiralty for measuring wave height, length, and period was of the hydrostatic pressure type. The instrument was located on or near the sea-bed and was connected by submarine cable with electro-visual or electro-photographic recording instruments on shore. An instrument of this type has the advantage of being easy to lay and comparatively robust, but it suffers from the disadvantage of recording the average of pressure over an area and not the instantaneous pressure corresponding to the hydrostatic head immediately above the instrument. To some extent that disadvantage was overcome by adjustment of the constants of the electric circuit of the instrument and by the employment of elaborate calibration curves, but the difficulty remained, though in a lessened degree, of recording simultaneously both long and short waves superimposed on each other. It was, therefore, decided to develop additional instruments for recording the wave-height by direct measurement and one type, which was employed in the trials with considerable success, consisted of a fixed mast, about 70 feet high, erected on the sea bed and having attached to it a vertical row of watertight float switches spaced at 6-inch intervals. Those switches were arranged to be operated by any rise or fall of the water-level at the mast, and the operation of the switch in turn varied the resistance and current in an electric circuit. By those comparatively simple means a direct recording, accurate to within 6 inches, was obtained on a time-base diagram of the passage of each individual wave. Two identical masts were used, one being located outside and one within the trial breakwater. Arrangements were also installed for measuring the rate of travel of an individual wave front so that the diagrams obtained from the two instruments could be synchronized and an actual figure of reduction on an individual wave obtained.

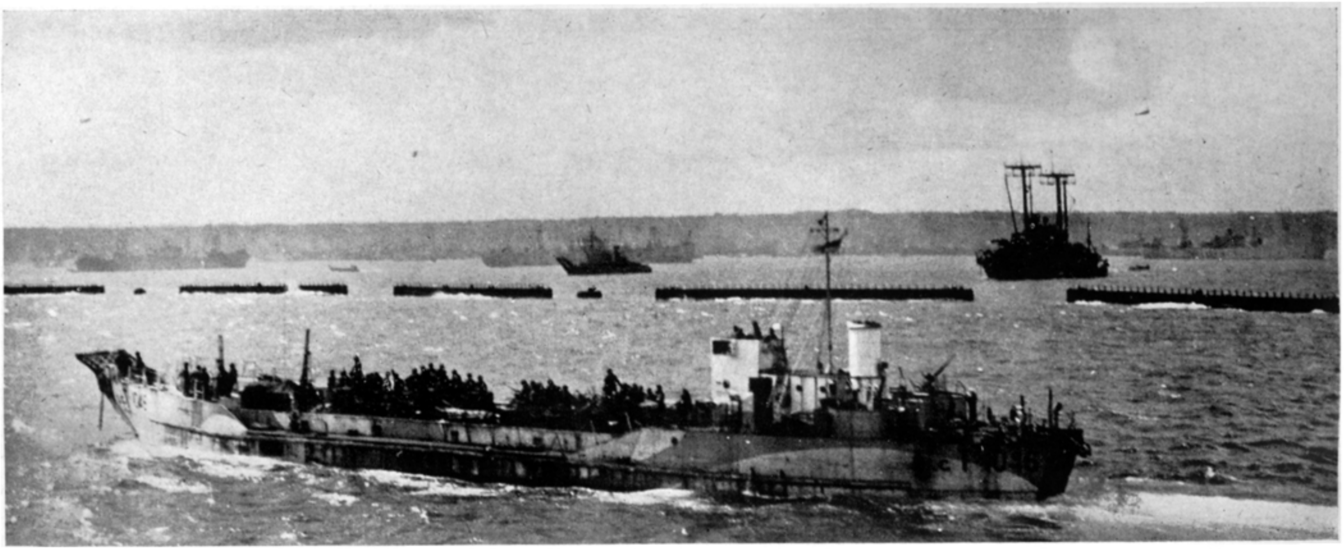
The development of all those new instruments and their installation, trial, and final adjustment had, of course, to proceed concurrently with the other work of development.

The observation post was also equipped with wind-speed recorders and the usual meteorological instruments, and was manned 24 hours a day from the commencement of the full-scale trials in February 1944.

On the 1st and 2nd April an onshore gale was recorded with a wind strength of force 7 gusting up to force 8 resulting in a sea up to 170 feet long and 8 feet high. That sea corresponded to a stress on the breakwater of approximately double that resulting from the originally estimated sea of 8 feet high and 100 feet long. Under those conditions the floating harbour proved to be completely successful. The waves were reduced in the lee of the breakwater to approximately 2 feet in height. The effect on vessels sheltering in the lee of the breakwater was more marked than even those figures indicate. For example, during the passage from Portland to the floating harbour, a U.S.N. mine-sweeper rolled her scuppers under on several occasions and, when beam on to the sea, it was impossible to walk about the decks without holding on. In the lee of the breakwater it was possible to lower and board a small boat and row about and reboard the mine-sweeper without difficulty. A picture of the breakwater during that gale is shown in ***Fig. 6*** (facing p. 265).

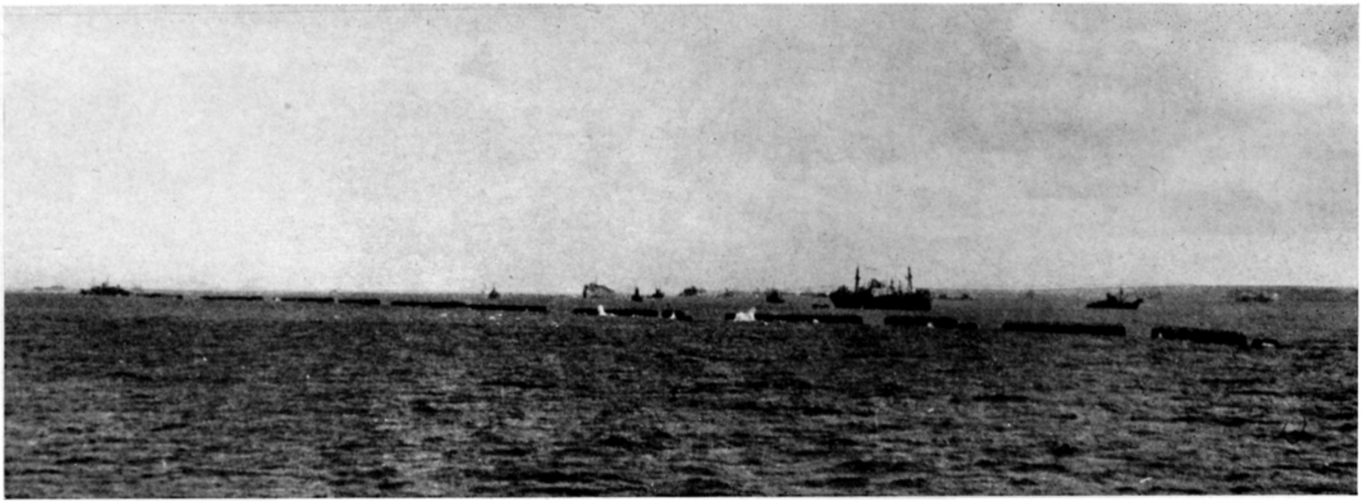
The opportunity afforded by those trials was also taken to test out various alternative modes of coupling Bombardon units to themselves

Fig. 7 (a).



BOMBARDON AT ST. LAURENT.

Fig. 7 (b).



Bombardon at Arromanches.

and to their mooring buoys. Between the majority of units a coupling, consisting of twin 18-inch cable-laid manilla rope, was employed. Each link had an eye splice at each end and the twin links were divisible half-way between units by means of pins and shackles. That form of coupler was successful and was adopted with one modification in the final operational assembly. The modification was to form the twin manilla into a strop instead of two separate links with individual eye-splices. In any future designs, however, where a strop is used, it will be desirable to use a strop thimble having a much more gentle lead than normal in order to lessen the stress on the strop lashing just below the thimble.

Chain links and a special form of spring shock-absorbing coupling were also employed but were found to have insufficient give and were abandoned. Had more time for development existed, there is no doubt, however, that a satisfactory form of all-metal flexible coupler would have been produced.

On frequent occasions during the trials, the sea in Weymouth bay, which at the best of times is not noted for its smoothness, was unsuitable for working small scraft. Yet on no occasion during the 3½ months of the trials was such work prevented in the lee of the floating breakwater, while on a number of occasions delicate work on instruments involving almost complete absence of motion was successfully accomplished.

Operation "Neptune".

The naval aspect of operation "Overlord" was known as operation "Neptune". As part of the operation, the first sections of the floating breakwater sailed with the invasion fleet on D-day. The units were towed in pairs at 50-foot spacing, the same manilla couplings which served to secure them to the mooring buoys also serving as towing links between the pairs of units. Towing proceeded without difficulty in seas up to 7 feet high and 200 feet long.

By D-day + 2 the first lengths of floating breakwater were providing shelter off the French coast. Both floating harbours were completed as single-line breakwaters with but one hitch by D-day + 6. The one hitch proved to be that both floating breakwaters were found to be moored in 11 to 13 fathoms, whereas, of course, the moorings had been designed for the same depth as the tests, namely, 7 fathoms. In the first fortnight the combination of blockships and floating breakwaters provided practically all the sheltered water used by the invading armies. During that stormy and critical period a great army of men and a vast quantity of stores was successfully landed with the help of that shelter, and a supply position was established on shore sufficient to secure bridgeheads against any attacks the enemy were in a position to launch against them at that time. The floating breakwater at St. Laurent is shown in *Fig. 7* (*a*), and that at Arromanches in *Fig. 7* (*b*).

A test was made at Arromanches on the 15th June and instrument readings then showed that, with a wind strength of force 5, the breakwater reduced the height of the waves by the predicted amount and the maximum height of sea inside the breakwater was less than 18 inches. On 19th June commenced the worst gale experienced in the English Channel in June for over 40 years.

During the 4 days from the 19th to the 23rd, seas over 15 feet high and 300 feet long drove in on the two Mulberries. *The stresses generated by those great waves were more than eight times those with which the harbour components were originally designed to compete.*

No fair-minded structural engineer would condemn a structure because it could not withstand stresses many times greater than those for which it was designed, and no one would blame those in authority, who had the difficult task of settling the maximum dimensions of waves for which these harbours were designed, for not taking into account, in such an operation, the conditions of a gale which had not occurred in the summer months in that part of the world during the last 40 years.

At Saint Laurent, where all the components were equally exposed, the blockships received such a battering that all of them either sank into the sand, partially turned over, or broke their backs. Even the battleship "Centurion" suffered the latter fate. Of the Phoenix units, twenty-five of the twenty-eight units exposed to the sea disintegrated. Under those circumstances the fact that the floating breakwater continued to function for over 30 hours before a single unit failed is a most noteworthy result. Thereafter the gale made a clean sweep, and when it subsided only those units of the two Mulberries which had been sheltered by the Calvados Reef remained unharmed and unmoved.

In the immediate shadow cast by the storm, many theories were advanced to explain the destruction of a large part of the harbour equipment. Most of those theories in the calmer light of retrospective scrutiny, may now be labelled as secondary contributary factors whilst a few have been proved to have been completely unfounded. In the latter category, the Authors are happy to state, may be placed the theory that any of the harbour suffered damage from drifting Bombardons. In the words of the official Admiralty report, "the suggestion that Phoenix units collapsed because they were hit by drifting Bombardons was proved to be incorrect". The overriding cause of the destruction of all the various components of the Mulberry harbours which were exposed to the full force of the gale was the fact that they were subjected to stresses far in excess of their designed capacity.

But, despite the destruction wrought by the gale, the floating breakwaters had performed a valuable function during the immensely important build-up period and in the words of the official report on the operation :—

"A full-scale breakwater, assembled off the Dorset coast in April 1944, successfully withstood an on-shore gale of force 7 (30 m.p.h.) with gusts up to force 8 (39 m.p.h.).

"The Bombardons were towed across the Channel without incident and according to programme. They arrived in position on D + 2 day and substantial lengths were in position by D + 3 day. The floating breakwater at the American harbour was completed by D + 6 and was sheltering large numbers of ships. The floating breakwater at the British harbour was completed by D + 7. Each breakwater was one mile in length, and consisted of 24 units.

"On several occasions after D-day the breakwaters withstood winds of the strength for which they were designed, namely force 6.

"Both breakwaters were moored in 11–13 fathoms, giving sufficient depth inshore for Liberty ships to anchor. In this depth they reduced the height of the waves by the measured amount of 50%, which represents a 75% reduction in wave energy. These measurements were carefully made at the British harbour on the 16th June, 1944 with a wind blowing force 5–6. Unloading operations and small boat work were going on inside the breakwater at that time which would not have been possible outside the breakwater.

"The staff requirements were, therefore, substantially achieved and with some margin on the right side."

Speaking of the results achieved, the report continues :—

"One of the most essential features of the OVERLORD plan was a rapid building up of troops and materials onshore during the first 14 days. On this depended the Allied ability to meet an enemy counter offensive. Of the three breakwater components, the blockships were in position first, to be followed within two days by the Bombardons. The Phoenix units did not arrive in quantity until several days later. The weather in the first fortnight was bad, and on a number of occasions the wind blew force 5–6 and the sea was rough. During this initial and very critical period both blockships and floating breakwaters played their part by sheltering hundreds of craft, and their presence enabled many operations to take place which would otherwise have been impossible."

Of the gale itself, the report says :—

"The floating breakwaters at both harbours withstood about 30 hours of this gale before serious damage occurred. This is impressive."

CONCLUSIONS.

The floating-harbour principle, briefly outlined in this Paper, is one of considerable importance. In the Authors' opinion, it has a very wide application in the science of harbour engineering. Careful estimates show that in many places where, on account of high first cost, the nature of the bottom, or the depth of the water, a fixed harbour is out of the question, a floating harbour will provide a satisfactory and permanent solution.

The advantages of this method of construction are many. In the normal case the cost of the floating type of harbour is of the order of one-fifth to one-twentieth of the fixed type. The area of sheltered water may be readily enlarged by the addition of units and the mere re-siting of moorings. There is no interference with the local underwater currents and, in consequence, a complete freedom from silting and scour.

A floating harbour may be erected in a matter of weeks whereas the fixed harbour is usually months or even years in the erection phase of construction.

A floating harbour may be arranged to cater with a seasonal or temporary trade or requirement and the units then moved on or stored away in the off-season. In this way a temporary floating breakwater can be utilized to protect fixed harbour works during the erection phase.

Although, when first suggested, the floating breakwater appeared as a rather startling innovation, and although the sceptics found many reasons before its trial why it could not work, yet, in fact, it did work and, inside its designed capacity, it successfully accomplished its allotted task in the invasion and liberation of Europe.

The Paper is accompanied by the following Appendix, and by two drawings and eleven photographs, from which the half-tone page plates and the Figures in the text have been prepared.

APPENDIX.

Mathematical Theory.

The following is an approximate treatment of the problems of surface waves and their action on floating objects. The examination is carried only so far as is necessary to provide an approximate solution of the problems of "Lilo" and "Bombardon" design.

The Mechanics of Surface Waves.

The wave theory given in this section is mostly to be found in any of the standard works on hydrodynamics. It is repeated here for ease of reference. The expressions given are, for the most part, those for surface waves in which the wave-height-to-length ratio is small.

Let a frictionless fluid be contained between the two parallel vertical sides of a horizontal canal. Let these sides be unit distance apart and stretch to $x = +\infty$ and $x = -\infty$. Also let :—

H denote depth of fluid in canal,
σ „ angular velocity,
ρ „ density of fluid,
a „ amplitude (half total movement),
λ „ wavelength,

$$K = \frac{2\pi}{\lambda},$$

$$g = 32{\cdot}16,$$

x, y = co-ordinates of a particle in its undisplaced position,
$x + X, y + Y$ = co-ordinates of a particle in its displaced position,
and t denote time in seconds.

Then, as developed by Airey, the expressions for the co-ordinates of a particle of the fluid when acted upon by a system of uniform travelling waves moving from $x = +\infty$ to $x = -\infty$ are

$$X = a\frac{\cosh K(y + H)}{\sinh KH}\cos(Kx - \sigma t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$Y = a\frac{\sinh K(y + H)}{\sinh KH}\sin(Kx - \sigma t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Particles whose displacement satisfy these equations move in elliptical orbits. The major axis of the ellipse is horizontal and is equal to

$$2a\frac{\cosh K(y + H)}{\sinh KH}$$

while the minor axis is vertical and equal to

$$2a\frac{\sinh K(y + H)}{\sinh KH}$$

These are the expressions for shallow-water waves.

Where $H > \frac{\lambda}{2}$, then e^{-KH} in the above expressions becomes small and the equations may be rewritten as :

$$X = ae^{Ky}\cos(Kx - \sigma t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and $$Y = ae^{Ky}\sin(Kx - \sigma t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Particles obeying equations (3) and (4) move in circular orbits of radius ae^{Ky}. If $x = 0$ in equations (3) and (4) they become :

$$X = ae^{Ky}\cos\sigma t \text{ and } Y = -ae^{Ky}\sin\sigma t$$

By giving x various values between 0 and λ, then for a given value of t the profile of the wave at that instant of time may be traced.
If expressions (3) and (4) are replaced by

$$X = ae^{Ky}\cos(Kx + \sigma t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and $$Y = ae^{Ky}\sin(Kx + \sigma t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

then a wave is obtained which travels in a negative direction, as opposed to the waves of (3) and (4) which travel in a positive direction. The particles in (5) and (6) move in an anti-clock-wise direction.

By making $y = 0$ in the above expressions for deep-water waves, the displacement co-ordinates are obtained for a particle at the free surface of the fluid.

The particles in these expressions (3) to (6) move in circular orbits with constant angular velocity equal to :

$$\sigma = \left(\frac{2\pi g}{\lambda}\right)^{\frac{1}{2}}$$

This angular velocity remains constant at all depths but the radius of the orbit diminishes rapidly as y increases. This fact is clear from the following Table in which the radius of the orbit of a particle in a deep water wave is given for various values of $\frac{y}{\lambda}$.

$\frac{y}{\lambda}$	0	0·05	0·10	0·15	0·20	0·25	0·30	0·40	0·50	0·60	0·80	1·00
Radius of orbit . .	1·0	0·73	0·53	0·39	0·28	0·21	0·15	0·081	0·043	0·023	0·0066	0·0019

The above expressions (3) to (6) are for a single system of travelling waves of uniform height and length. Similar equations will be found in standard works for a single uniform system of standing waves. For shallow water these are :

$$X = - a \frac{\cosh K(y + H)}{\sinh KH} \sin Kx \,.\, \sin (\sigma t + \omega) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and
$$Y = a \frac{\sinh K(y + H)}{\sinh KH} \cos Kx \,.\, \sin (\sigma t + \omega) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

By making x of such value that $\cos Kx = 1$ it is possible to determine the motion of a particle at the position along the X axis when all the particle movement is taking place up and down the Y axis. The expressions obtained are :

$$X = 0$$

and
$$Y = a \frac{\sinh K(y + H)}{\sinh KH} \sin (\sigma t + \omega)$$

If $H > \frac{\lambda}{2}$, expressions (7) and (8) simplify to :

$$X = - ae^{Ky} \sin Kx \,.\, \sin (\sigma t + \omega) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and
$$Y = ae^{Ky} \cos Kx \,.\, \sin (\sigma t + \omega) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

But expressions (9) and (10) may be obtained by combining expression (3) with (5) and (4) with (6), after allowing for the appropriate change of phase in the reflected wave.

A standing-wave system may therefore be considered as the combination of two travelling-wave systems of equal amplitude and wave-length, but moving in opposite directions. This is the condition which results from perfect reflexion of an incident wave striking a vertical barrier placed across the canal at 90 degrees to the direction of travel. If this barrier be regarded as the position $x = 0$ then the following relations hold.

For the incident wave travelling in the negative direction with amplitude $\frac{1}{2}a$:

$$X_1 = \tfrac{1}{2}ae^{Ky} \cos (Kx + \sigma t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and
$$Y_1 = \tfrac{1}{2}ae^{Ky} \sin (Kx + \sigma t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

For the reflected wave travelling in the positive direction with amplitude $\frac{1}{2}a$:

$$X_2 = - \tfrac{1}{2}ae^{Ky} \cos (Kx - \sigma t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and
$$Y_2 = - \tfrac{1}{2}ae^{Ky} \sin (Kx - \sigma t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Combining these two systems :

$$\begin{aligned} X = X_1 + X_2 &= \tfrac{1}{2}ae^{Ky}\{\cos (Kx + \sigma t) - \cos (Kx - \sigma t)\} \\ &= - ae^{Ky} \sin Kx \,.\, \sin \sigma t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15) \end{aligned}$$

and
$$\begin{aligned} Y = Y_1 + Y_2 &= \tfrac{1}{2}ae^{Ky}\{\sin (Kx + \sigma t) - \sin (Kx - \sigma t)\} \\ &= ae^{Ky} \cos Kx \,.\, \sin \sigma t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16) \end{aligned}$$

Expressions (15) and (16) are identical with (9) and (10), the term ω having been eliminated by placing the barrier at the position $x = 0$.

This method of combining two systems of travelling waves may also be applied to the case when the amplitudes are unequal. Let the amplitude of the incident wave be a_1 and that of the reflected wave a_2. Then for the incident wave :

$$X_1 = a_1e^{Ky} \cos (Kx + \sigma t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and
$$Y_1 = a_1e^{Ky} \sin (Kx + \sigma t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and for the reflected wave :

$$X_2 = - a_2e^{Ky} \cos (Kx - \sigma t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and
$$Y_2 = - a_2e^{Ky} \sin (Kx - \sigma t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Combining these equations gives the following expressions for the displacement co-ordinates of a particle subjected simultaneously to the two systems :

$$X = X_1 + X_2 = a_1e^{Ky} \cos (Kx + \sigma t) - a_2e^{Ky} \cos (Kx - \sigma t) \quad . \quad . \quad (21)$$

and
$$Y = Y_1 + Y_2 = a_1e^{Ky} \sin (Kx + \sigma t) - a_2e^{Ky} \sin (Kx - \sigma t) \quad . \quad . \quad (22)$$

These are the equations for elliptic harmonic motion and indicate that the orbit of the particle is an ellipse whose major axis is $2e^{Ky}(a_1 + a_2)$, and whose minor axis is $2e^{Ky}(a_1 - a_2)$. At the point $x = 0$ these equations reduce to:

$$X = e^{Ky}(a_1 - a_2) \cos \sigma t$$
$$Y = e^{Ky}(a_1 + a_2) \sin \sigma t,$$

indicating that at this point the major axis of the ellipse lies along the Y axis.

If $x = \frac{\lambda}{4}$ then equations (21) and (22) reduce to:

$$X = - e^{Ky}(a_1 + a_2) \sin \sigma t$$

and
$$Y = e^{Ky}(a_1 - a_2) \cos \sigma t,$$

indicating that at this point the major axis of the ellipse lies along the X axis. At other points the major axis makes an angle Kx with the Y axis.

The movement of the particle round the ellipse is anti-clockwise, indicating a resultant wave travelling in the negative direction, that is, in the same direction as the incident wave.

The energy contained in one wave of a single uniform system of travelling waves of the type in expressions (3) to (6) is $\frac{1}{2}g\rho a^2\lambda$ per unit length of wave front. That contained in two such systems each of amplitude $\frac{1}{2}a$ and wave-length λ is $\frac{1}{4}g\rho a^2\lambda$. This also gives the energy in a standing-wave system resulting from perfect reflexion.

The rate of transmission of energy in a travelling-wave system of amplitude a is $\frac{1}{2}g\rho a^2 V$, where V is the group velocity which, in the case of deep water waves, is one half the wave velocity. It follows that, if the incident wave is partly reflected and partly transmitted by the breakwater, without loss of energy, then

$$a_1^2 = a_2^2 + a_3^2 \quad \ldots \ldots \ldots \quad (23)$$

If the energy is dissipated at the breakwater at the rate of R ft.-lb. per second, then

$$a_1^2 = a_2^2 + a_3^2 + \frac{2R}{\rho V} \quad \ldots \ldots \ldots \quad (23a)$$

(1) *Theory of Floating Breakwaters.*

In the previous section expressions were developed for waves travelling between the parallel sides of an infinitely long canal. If a rigid vertical wall is erected across such a canal at right angles to the sides and reaching to the bottom, then any wave travelling along the canal will experience total reflexion on striking the wall. Assuming that the incident waves travel in the negative direction, then the reflected waves will travel in the positive direction with equal wave-length and amplitude to those of the incident waves. These two wave systems will combine to produce the standing-wave systems of expressions (7) and (8).

If the depth $H > \frac{\lambda}{2}$, equations (7) and (8) may be replaced by those given at (15) and (16).

Now if, instead of extending to the bottom of the canal, the barrier finishes at a depth $y = - D$ then part of the incident wave energy will pass under the barrier and the remainder will be reflected. Assuming $H > \frac{\lambda}{2}$ the amplitude of the particles at depth D will be approximately ae^{-KD} and the energy passing under the barrier will be:

$$\tfrac{1}{2}g\rho a^2\lambda e^{-2KD} \quad \ldots \ldots \ldots \quad (24)$$

This energy will reappear eventually as a travelling-wave system on the lee side of the barrier. This wave system will have the same wave-length as the original incident wave and will move in the negative direction with a surface amplitude of approximately:

$$ae^{-KD}. \quad \ldots \ldots \ldots \quad (25)$$

The energy in the reflected wave will be:

$$\tfrac{1}{2}g\rho a^2\lambda - \tfrac{1}{2}g\rho a^2\lambda e^{-2KD} = \tfrac{1}{2}g\rho\lambda a^2(1 - e^{-2KD}) \quad \ldots \quad (26)$$

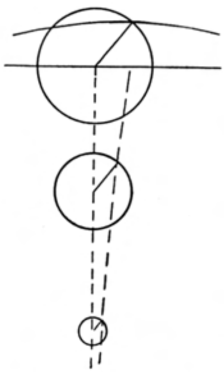
and the amplitude of the reflected wave at some distance from the barrier will be:

$$a(1 - e^{-2KD})^{\frac{1}{2}} \quad \ldots \ldots \ldots \quad (27)$$

In the case just considered it was assumed that the barrier was rigid or had infinite mass. But floating barriers or breakwaters, of necessity, have finite mass in themselves and, unless they are rigidly constrained, they will move with the wave motion to a greater or lesser degree, dependent in part on the mass of the barrier. Before considering the effect of mass it will be instructive to look briefly at the movement of a barrier having no mass.

Fig. 8 shows the orbits of particles at depths $y = 0$, $0{\cdot}1\lambda$ and $0{\cdot}2\lambda$. The particles are shown at time $\sigma t = 60$ degrees, whilst the mean position of the particles is assumed to be at $Kx = 0$. Through the particles has been drawn, in section, a flexible membrane of zero mass extending to infinite depth. Such a barrier will follow the movement of the particles and the whole of the energy of the incident wave will be reproduced as a transmitted wave on the left hand side of the barrier. The amplitude will remain the same on both sides and there will be no reflexion. Furthermore, this condition will be the same in a frictionless fluid, even where the motion of the membrane is along the X-axis only and no motion takes place along the Y-axis. Suppose that, in place of the barrier in *Fig. 8,* a body is employed having a mass m per unit volume. Then, in addition to other forces operating on the particles, a force will be introduced by each element of mass which will be proportional to its acceleration at any instant. If the motion of the barrier along the X-axis is simple-harmonic, the acceleration arising therefrom will be a maximum at the extreme limits of movement of the barrier. At this instant the barrier will be stationary and the force producing this acceleration arises from an unbalance of the forces produced by the water on either side of the barrier. But the wave on the left-hand side of the barrier is being produced by the motion of the barrier. At the instant when the barrier is at the extreme limit of travel, the particles on the left or transmitted-wave side must be passing through their mean position on the Y-axis. The water on this side at this instant must be at mean level and the motion of the particles along the X-axis must be zero.

Fig. 8.



PARTICLE MOTION AT VARIOUS DEPTHS.

At this instant the water-level on the incident-wave side of the barrier must be different from that on the transmitted-wave side. By considering various such positions through a wave cycle it will become apparent that the motion of the particles on the incident-wave side are following an orbit which resembles an ellipse. But elliptical orbits indicate the presence of two systems of travelling waves of equal wave-length and unequal amplitude travelling in opposite directions. This will be found to be the effect of adding mass to the barrier. The motion of the particles on the incident-wave side of the barrier becomes elliptical, some part of the incident wave energy being reflected and some transmitted by the barrier.

(2) *Wave Reflexion.*

Returning now to equations (21) and (22), and assuming for the moment that the floating barrier extends to infinite depth, then the following displacement co-ordinate expressions may be obtained :—

For the incident wave :

$$X_1 = a_1 e^{Ky} \cos (Kx + \sigma t)$$
$$Y_1 = a_1 e^{Ky} \sin (Kx + \sigma t) ;$$

for the reflected wave :

$$X_2 = -\, a_2 e^{Ky} \cos (Kx - \sigma t)$$
$$Y_2 = -\, a_2 e^{Ky} \sin (Kx - \sigma t) ;$$

and for the resultant wave :

$$X_R = X_1 + X_2 = a_1 e^{Ky} \cos (Kx + \sigma t) - a_2 e^{Ky} \cos (Kx - \sigma t)$$
$$Y^R = Y_1 + Y_2 = a_1 e^{Ky} \sin (Kx + \sigma t) - a_2 e^{Ky} \sin (Kx - \sigma t)$$

The expressions for X_R and Y_R may be rewritten as

$$X_R = e^{Ky}\{(a_1 - a_2)\cos Kx \,.\, \cos \sigma t - (a_1 + a_2)\sin Kx \,.\, \sin \sigma t\}$$
$$= e^{Ky}\sqrt{(a_1 + a_2)^2 \sin^2 Kx + (a_1 - a_2)^2 \cos^2 Kx} \,.\, \cos(\sigma t + \alpha_1) \quad . \quad . \quad (28)$$

where $$\tan\alpha_1 = \frac{(a_1 + a_2)\sin Kx}{(a_1 - a_2)\cos Kx} = \frac{a_1 + a_2}{a_1 - a_2}\tan Kx$$

and $$Y_R = e^{Ky}\{(a_1 - a_2)\sin Kx \,.\, \cos \sigma t + (a_1 + a_2)\cos Kx \,.\, \sin \sigma t\}$$
$$= e^{Ky}\sqrt{(a_1 + a_2)^2 \cos^2 Kx + (a_1 - a_2)^2 \sin^2 Kx} \,.\, \sin(\sigma t + \alpha_2) \quad . \quad (29)$$

where $$\tan\alpha_2 = \frac{a_1 - a_2}{a_1 + a_2}\tan Kx.$$

The displacement co-ordinates of the particles on the transmitted wave side of the barrier will be those of a single travelling-wave system of amplitude $a_3 = X_R$ since the movement along the X-axis is common to both sides.

Hence $$(a_1 + a_2)^2 \sin^2 Kx + (a_1 - a_2)^2 \cos^2 Kx = a_3^2 \quad . \quad . \quad . \quad . \quad (30)$$

From the energy equations :

$$a_3^2 = a_1^2 - a_2^2$$

Substituting and resolving :

$$\cos 2Kx = \pm\sqrt{1 - \left(\frac{a_3}{a_1}\right)^2} = +\frac{a_2}{a_1}.$$

If $\dfrac{a_3}{a_1} = 0$ in this expression, then $\cos 2Kx = \pm 1$ and $2Kx$ must equal 0, π, 2π, etc. Kx must therefore equal 0, $\frac{1}{2}\pi$, π, etc. But when $\dfrac{a_3}{a_2} = 0$, perfect reflexion is taking place and the above values of Kx all make $a_1\cos(Kx + \sigma t) - a_2\cos(Kx - \sigma t)$ zero, that is to say, the horizontal motion at the barrier is zero. These values of Kx therefore correspond to a rigid barrier rigidly fixed in position. The reflexion obtained from a Bombardon approximates to the above, but is somewhat less, because the Bombardon is not rigidly fixed. It is represented more closely by taking a value of Kx between 0 and $\dfrac{\pi}{4}$ (or more conveniently between $-\dfrac{\pi}{4}$ and 0. It can be seen that any range of values of Kx of length $\dfrac{\pi}{4}$ will cover all possible cases satisfying the imposed conditions.

Returning to the expression for α_1 in equation (28) this may be written as

$$\tan\alpha_1 = \frac{1 + \dfrac{a_2}{a_1}}{1 - \dfrac{a_3}{a_1}} \cdot \frac{\sin Kx}{\cos Kx}$$

but $$\cos 2Kx = \pm\sqrt{1 - \left(\frac{a_3}{a_1}\right)^2} = +\frac{a_2}{a_1}.$$

Substituting and resolving

$$\alpha_1 = \pm\tfrac{1}{2}\pi - Kx$$

When the angle Kx is positive, the positive value of $\frac{1}{2}\pi$ is employed and vice versa.

Let $$X_{RM} = \sqrt{(a_1 + a_2)^2 \sin^2 Kx + (a_1 - a_2)^2 \cos^2 Kx} \quad . \quad . \quad . \quad (31)$$

and $$Y_{RM} = \sqrt{(a_1 + a_2)^2 \cos^2 Kx + (a_1 - a_2)^2 \sin^2 Kx} \quad . \quad . \quad . \quad (32)$$

then the displacement co-ordinates for a particle on the incident-wave side of the barrier, and for negative values of Kx are :

$$X_R = X_{RM}e^{Ky}\cos(\sigma t - \tfrac{1}{2}\pi - Kx) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

and $$Y_R = Y_{RM}e^{Ky}\sin(\sigma t + \alpha_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

But since the X-axis component is common to both the incident- and transmitted-wave sides of the barrier, and represent in effect the movement of the barrier, the displacement co-ordinates for the transmitted wave must be :

$$X = X_{RM}e^{Ky}\cos(\sigma t - \tfrac{1}{2}\pi - Kx) \quad . \; . \; . \; . \; . \; . \; . \quad (35)$$

and

$$Y = Y_{RM}e^{Ky}\sin(\sigma t - \tfrac{1}{2}\pi - Kx) \quad . \; . \; . \; . \; . \; . \; . \quad (36)$$

These various relations are shown diagrammatically for a negative value of Kx in *Fig. 9.*

The following table gives various values of $\frac{a_3}{a_1}$, $\frac{a_2}{a_1}$, Kx, $\frac{x}{\lambda}$, and the angle of lag between the incident wave a_1 and the transmitted wave a_3

$\frac{a_3}{a_1}$	$\frac{a_2}{a_1}$	$-Kx$: degrees.	$\frac{x}{\lambda}$	Angle of lag : degrees.
0·1	0·99	3	0·0081	$84\frac{1}{4}$
0·2	0·98	$5\frac{3}{4}$	0·016	$78\frac{1}{2}$
0·3	0·95	$8\frac{3}{4}$	0·024	$72\frac{1}{2}$
0·4	0·92	$11\frac{3}{4}$	0·033	$66\frac{1}{2}$
0·5	0·87	15	0·042	60
0·6	0·80	$18\frac{1}{2}$	0·051	$53\frac{1}{4}$
0·7	0·71	$22\frac{1}{4}$	0·062	$45\frac{1}{2}$
0·8	0·60	$26\frac{1}{2}$	0·074	$36\frac{3}{4}$
0·9	0·44	32	0·089	26
1·0	0	45	0·126	0

(3) *The Pressure on the Barrier.*

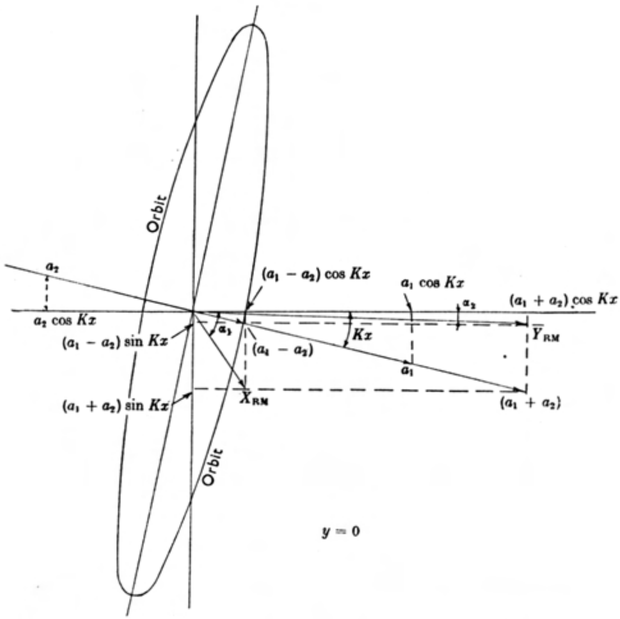
The next step is to examine the forces on the barrier which result from the hydrostatic pressure exerted by the moving water. They will consist of two parts, (*a*) a static part which would be present if the water were at rest and (*b*) a part depending on the motion of the water. To determine the total hydrostatic pressure it is necessary first to form the equations of motion of a small element of the water. In these calculations it is first supposed that the height $2a$ of the incident wave is small compared with its length (in marine waves the ratio $\frac{2a}{\lambda}$ is seldom as great as 0·1), and the calculations will be carried as far as the first order terms in $\frac{a}{\lambda}$ only. Where necessary the calculations are later extended to cover second-order terms.

Consider a small elementary prism of the fluid of unit length and rectangular cross-section $\delta x \, . \, \delta y$, centred at (x, y). If the pressure at (x, y) is $p(x, y)$, the pressure at any neighbouring point $(x + \delta x, y + \delta y)$ is $p + \frac{\partial p}{\partial x} \, . \, \delta x + \frac{\partial p}{\partial y} \, . \, \delta y$ to the first order.

Hence the resultant thrusts on the four lateral faces of the prisms are as in *Fig. 10.* These lead to the resultant forces $\frac{\partial p}{\partial x} \, . \, \delta x \, . \, \delta y$ in the horizontal negative X-direction and $\frac{\partial p}{\partial y} \, . \, \delta x \, . \, \delta y$ in the (downward) negative Y-direction. The other force acting on the prism is its weight $\rho g \delta x \, . \, \delta y$ (in absolute units) vertically downwards. The resultant of these forces must equal the mass-times-acceleration of the fluid at (x, y). If (X, Y) are the displacement co-ordinates from the undisturbed position of the fluid which is at (x, y) at time t, then *to the first order of small quantities* in X, Y, the acceleration at (x, y) has components $\ddot{X}$, $\ddot{Y}$. Hence the equations of motion of the elementary prism are, on cancelling $\delta x \, . \, \delta y$:

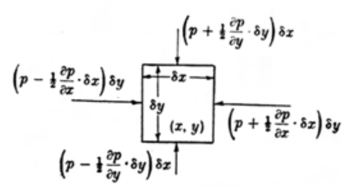
$$\frac{\partial p}{\partial x} = -\rho \ddot{X}$$

Fig. 9.



PARTICLE MOTION AT SURFACE OF A PARTIAL REFLECTOR.

Fig. 10.



PRESSURE DISTRIBUTION.

and $$\frac{\partial p}{\partial y} + \rho g = -\rho \ddot{Y}$$

Now, by hypothesis, X and Y are simple-harmonic functions of t with periods $\frac{2\pi}{\sigma}$; hence $\ddot{X} = -\sigma^2 X$ and $\ddot{Y} = -\sigma^2 Y$, and the above equations become:

$$\frac{\partial p}{dx} = \rho\sigma^2 X \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

and $$\frac{\partial p}{\partial y} + \rho g = \rho\sigma^2 Y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Integrating the second of these equations gives

$$p + p_0 = -\rho g y + \rho\sigma^2 \int Y dy + F$$

where F is independent of y. But for all the motions considered (see equations (7)–(22)):

$$\int Y dy = \frac{1}{K} Y;$$

and

$$\frac{dy}{dx} = Kx.$$

Hence, on comparing with equation (37), F must also be independent of x, and may be identified with the pressure, say p_0, at the free surface. Hence, to the first order, the pressure at any point (x, y) in the moving fluid is given by

$$p - p_0 = -\rho g y + \frac{\rho\sigma^2}{K} Y.$$

Writing p for $p - p_0$ (the excess pressure) and $\sigma^2 = Kg$, gives:

$$p = -\rho g y + \rho g Y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

To obtain the pressure against the barrier at any depth y on the incident wave side, it is only necessary to substitute:

$$Y_R = e^{Ky}\{a_1 \sin(Kx + \sigma t) - a_2 \sin(Kx - \sigma t)\}$$

for Y in expression (39).

If the barrier extends to a depth D, the total pressure on the incident-wave side is:

$$\int_{-D}^{Y_{RD}} p \, . \, dy = \rho g \left[-\frac{y^2}{2} + \frac{1}{K} e^{Ky}\{a_1 \sin(Kx + \sigma t) - a_2 \sin(Kx - \sigma t)\} \right]_{-D}^{Y_{RD}}$$

$$= \rho g \left[\frac{D^2}{2} + \frac{1}{K}(1 - e^{-KD})\{a_1 \sin(Kx + \sigma t) - a_2 \sin(Kx - \sigma t)\} \right] \quad . \quad (40)$$

on neglecting second-order terms.

Similarly the total pressure on the transmitted-wave side is, from (36), equal to:

$$\rho g \left[\frac{D^2}{2} + \frac{1}{K}(1 - e^{-KD}) a_3 \sin\left(\sigma t - \frac{\pi}{2} - Kx\right) \right] \quad . \quad . \quad . \quad . \quad (41)$$

Hence the resultant horizontal force on the breakwater due to the wave pressures is

$$\frac{\rho g}{K}\left(1 - e^{-KD}\right)\left\{a_1 \sin(Kx + \sigma t) - a_2 \sin(Kx - \sigma t) - a_3 \sin\left(\sigma t - \frac{\pi}{2} - Kx\right)\right\}$$

in the negative-X direction.

Now, since $X_3 = X_R = X_1 + X_2$, it follows that:

$$a_3 \cos\left(\sigma t - \frac{\pi}{2} - Kx\right) = a_1 \cos(Kx + \sigma t) - a_2 \cos(Kx - \sigma t)$$

for all values of t. Hence, increasing t by $\frac{\pi}{2\sigma}$, gives:

$$-a_3 \sin\left(\sigma t - \frac{\pi}{2} - Kx\right) = -a_1 \sin(Kx + \sigma t) - a_2 \sin(Kx - \sigma t) \quad . \quad . \quad (41a)$$

On substituting in the above expression, the resultant horizontal force on the breakwater in the direction of X-decreasing is, to the first order, equal to:

$$P = -\frac{2\rho g}{K}(1 - e^{-KD}) a_2 \sin(Kx - \sigma t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

This result shows that the horizontal force has a simple-harmonic variation of the same period as that of the incident waves. Also it is exactly the same as if *only that portion of the incident wave which is reflected gives rise to any force on the breakwater*, the energy of the remaining portion passing through the barrier (as it were) without having any effect upon it.

(4) *The Steady Pressure.*

Equation (42) does not reveal any steady pressure to be taken up by the supports or moorings. This is because, for the wave motions so far considered, there is no net momentum in any direction. To determine the steady pressure which actually exists in practice, it is necessary to extend the whole of the above calculations to take into account the finite height of the waves. When this is done it is no longer possible to regard the motion of any particular particle of the water as made up of the simple circular motions so far considered. The extension of the calculations requires a much more elaborate investigation of the equations of motion than the above.

By means of this more elaborate treatment it can be shown that two "Stokes" waves (which have a net momentum in the direction of their motion) of the same wave-length but different wave-heights, and travelling in opposite directions, can be combined to give a motion which satisfies the condition of constant pressure at the free surface to a second, or higher, order of approximation. These two waves may be taken as representing the incident and reflected waves respectively at points which are not too close to the breakwater. A third "Stokes" wave will similarly represent the transmitted wave at points sufficiently removed from the breakwater. The resultant is a net average momentum on the incident-wave side of amount $\frac{\pi\sigma\rho}{K}(a_1^2 - a_2^2)$ per wave-length in the negative-X direction, and the transmitted wave has an average momentum of amount $\frac{\pi\sigma\rho^2}{K} . a_3^2$ per wave-length, in the same direction.

Hence the momentum destroyed per wave *period* by the breakwater (no matter what its precise mechanism may be) is $\frac{\pi\sigma\rho}{K}(a_1^2 - a_2^2 - a_3^2)$, and this requires an average force of amount :

$$F = \tfrac{1}{2}g\rho(a_1^2 - a_2^2 - a_3^2) \qquad \ldots \ldots \ldots \quad (43)$$

This interesting result shows that there will be *no steady pressure on the breakwater* (even when waves of finite height are considered) *unless there is dissipation of energy there.*

In general the energy per unit time received at the breakwater from the incident wave is $\frac{1}{4}g\rho a_1^2 V$, and the energy propagated away in the reflected and transmitted waves is $\frac{1}{4}g\rho(a_2^2 + a_3^2)V$, where V denotes the velocity of the waves.

Hence the energy dissipated at the breakwater per unit time is :

$$\tfrac{1}{4}g\rho V(a_1^2 - a_2^2 - a_3^2) \qquad \ldots \ldots \ldots \quad (44)$$
$$= \tfrac{1}{2}FV, \text{ from equation (41).}$$

This is in agreement with the fact that the *group* velocity of deep-water waves is $\frac{1}{2}V$.

(5) *The Motion of a Rigid Floating Barrier.*

In order to simplify the next stage of the treatment the following assumptions are made :—

(*a*) That the height of the incident wave does not exceed 0·1 of the length.
(*b*) That D is not less than 0·15.
(*c*) That H exceeds 0·5.

Equation (24) gives the energy passing under a fixed barrier of depth D. The ratio of this energy to that of the incident wave is $\frac{1}{e2KD}$. If $\frac{D}{\lambda}$ is equal to 0·15, then this ratio is 0·15, whilst if $\frac{D}{\lambda} = 0{\cdot}2$, then the ratio $= 0{\cdot}08$. This energy reappears as a travelling wave at some distance behind the barrier. A similar effect takes place with the reflected energy given by equation (26). This reflected energy produces a reflected wave of amplitude $a(1 - e^{-2KD})^{\frac{1}{2}}$ at some distance from the barrier. But at the barrier the energy in the reflected wave is concentrated into the depth D and in consequence the amplitude of the reflected wave at the barrier will be substantially the same as that of the reflected wave produced by a barrier of infinite depth. The same result may be deduced for a barrier moving with amplitude a_3, except that the reflected wave will be reduced in amplitude by the motion of the reflecting barrier. The effect of giving the barrier finite depth is to leave the equations for the motion of a particle at the surface of the barrier substantially unaltered, but the total pressure against the barrier is reduced, by reason of its finite depth, and the amplitudes of the reflected and transmitted waves at a distance from the barrier are also reduced, by reason of the initial energy concentrated into depth D being finally distributed over infinite depth. Furthermore, the final transmitted wave will be the resultant of this transmitted energy plus the energy passing under the barrier.

It will now be necessary to consider the oscillation of a finite barrier. Equation (35) gives the motion along the X-axis of the particles of the transmitted wave. The elements of an ideal wave-maker would have to obey the requirements of this equation. The floating barrier under consideration is of course a wave-maker on its lee side. But a floating barrier with rigid sides will not obey expression (35). It will oscillate with too great an amplitude at some depths and too little at other depths. It will tend, however, to sweep out approximately the same volume of water as the ideal barrier, and will endeavour to store the same kinetic energy in the position of maximum velocity. A close approximation to the barrier required by expression (35) is to be found by taking a rigid plane sheet and pivoting it at a point where

$$y = -\frac{1}{Ke^{-\frac{KD}{2}}} = -\frac{\lambda}{2\pi e^{-\frac{KD}{2}}} = -\gamma$$

The amplitude of the barrier at $y = 0$ will be a_3, as before, whilst the amplitudes at other points will be $a_3\left(1 + \frac{y}{\gamma}\right)$.

The amplitude at depth $y = -D$ will be $a\left(1 - \frac{D}{\gamma}\right)$. This motion is, in part, a linear oscillation of the barrier along the X-axis and, in part, an angular rotation about its centre of mass. These expressions will be substantially correct provided $D < \frac{\lambda}{3}$.

A barrier will only oscillate about a point $y = -\gamma$ with an amplitude a_3 at point $y = 0$ if the centre of mass and the centre of pressure arising from wave action are a certain distance apart, and if the total effective mass and the total effective pressure equal certain quantities. These two further factors will now be investigated.

Fig. 11 shows a floating barrier of the type previously considered with depth D and pivoting about the point $y = -\frac{1}{Ke^{-\frac{KD}{2}}}$. The total linear amplitude at $y = 0$ is a_3. The maximum angular displacement for small values of θ_m will be :

$$\theta_m = \frac{a_3}{Ke^{-\frac{KD}{2}}} = \frac{a_3}{\gamma}.$$

The angular displacement at any time will be :

$$\theta = \frac{a_3}{\gamma} \sin \phi$$

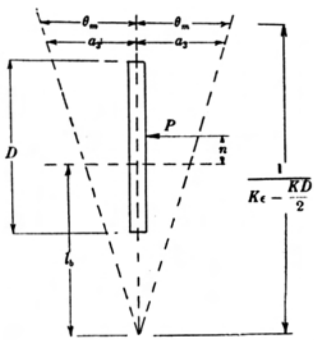
If l_G is the distance from the pivot-point to the centre of mass of the barrier the linear displacement of this point will be

$$\theta l_G = \frac{a_3 l_G}{\gamma} \sin \phi = \frac{a_3 l_G}{\gamma} \sin (\sigma t + \alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

where α denotes an angle whose value is not material for this part of the treatment.

The maximum acceleration at this point is $\frac{\sigma^2 a_3 l_G}{\gamma}$ and the torque required to accelerate this mass will be $\frac{\sigma^2 a_3 l_G}{\gamma} . M$, where M denotes the total effective mass of the barrier and the mass of the water undergoing acceleration on the transmitted-wave side. The subject of effective mass is dealt with in paragraph (7) (*post*). The maximum angular acceleration about the centre of mass is $\frac{\sigma^2 a_3}{\gamma}$, and if I denotes the effective moment of inertia of the barrier, including the mass of any water dragged

Fig. 11.



SIMPLIFIED SECTION OF A FLOATING BARRIER.

with the barrier, then the torque required to produce the required angular acceleration round the centre of mass is $\frac{\sigma^2 a_3}{\gamma} I$.

Now if, in *Fig. 11*, P denotes the resultant wave-pressure, and n is the distance between P and the centre of mass, then the following relations may be established :—

$$\frac{\sigma^2 a_3 l_G}{\gamma} M = P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

and

$$\frac{\sigma^2 a_3}{\gamma} I = Pn = T_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Hence, if R_G is the radius of gyration of the barrier about its centre of mass, then

$$n = \frac{R_G{}^2}{l_G} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

If the mass distribution is such that the above relations are satisfied then the barrier will pivot about point $y = -\gamma$ with amplitude a_3. If n is greater than that given by equation (44), then the barrier will oscillate with greater angular amplitude and the wave generated on the transmitted-wave side will be greater in consequence. Where the whole mass is concentrated at the bottom of the barrier, the transmitted wave will be almost as great as the incident wave and there will be practically no reflexion. On the other hand, where n is less than that given by equation (48), the barrier will tend to oscillate about its centre of mass with less angular amplitude than that calculated above. The efficiency of the barrier as a wave-maker will be reduced and the transmitted-wave amplitude will be reduced slightly. The reduction achieved by making n less than that calculated above is far less marked than the increase which results from making n greater than calculated from (48). This constant n is one of the fundamental factors in Bombardon design.

(6) *The Distribution of Pressure on the Breakwater, and the Line of Action of the Resultant Horizontal Thrust.*

To apply the results of the last section it is necessary to know the line of action of the resultant horizontal force on the breakwater. The magnitude of this force has already been found in paragraph (3) (*ante*).

The pressure at depth $-y$ on the incident-wave side of the breakwater is given by equation (39) as

$$p = -\rho g y + g Y_R$$

The clockwise moment, about the point $y = 0$, of the pressure on an elementary strip of unit breadth and depth δy is $\rho g(y - Y_R)y\delta y$, and the total moment about the water-line, $y = 0$, of the pressure on unit breadth of the breakwater on the incident-wave side is :

$$\rho g \int_{-D}^{Y_{RO}} (y - Y_R) y \, . \, dy$$

where Y_{RO} is the value of Y_R at $y = 0$, or :

$$Y_{RO} = a_1 \sin (Kx + \sigma t) - a_2 \sin (Kx - \sigma t) \quad . \quad . \quad . \quad . \quad (50)$$

This reduces, on retaining terms of the first order only in Y_{RO}, to

$$\rho g \left[\frac{D^3}{3} - \left\{a_1 \sin (Kx + \sigma t) - a_2 \sin (Kx - \sigma t)\right\}\left\{\frac{D}{K}e^{-KD} - \frac{1}{K^2}(1 - e^{-KD})\right\}\right] \quad (51)$$

On dividing by the total pressure on the incident-wave side, which is given by equation (40), it is found that the depth of the centre of pressure on this side is given by :

$$d_1 = \frac{\frac{2}{3}D + 2\left\{\frac{1}{K^2D^2}(1 - e^{-KD}) - \frac{e^{-KD}}{KD}\right\}Y_{RO}}{1 + \frac{2}{KD^2}(1 - e^{-KD})Y_{RO}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Similarly it is found that the counter-clockwise moment, about $y = 0$, of the pressure, on the transmitted-wave side is :

$$\rho g \left[\frac{D^3}{3} - \left\{ a_3 \sin (\sigma t - \tfrac{1}{2}\pi - Kx) \right\}\left\{\frac{D}{K}e^{-KD} - \frac{1}{K^2}(1 - e^{-KD})\right\}\right] \quad . \quad . \quad (53)$$

and the depth of the centre of pressure on this side is therefore :

$$d_2 = \frac{\frac{2}{3}D + 2\left\{\frac{1}{K^2D^2}(1 - e^{-KD}) - \frac{e^{-KD}}{KD}\right\}a_3 \sin (\sigma t - \frac{1}{2}\pi - Kx)}{1 + \frac{2}{KD^2}(1 - e^{-KD})a_3 \sin (\sigma t - \frac{1}{2}\pi - Kx)} \quad . \quad . \quad (54)$$

Also the resultant clockwise moment about the water-line of the wave-pressures on both sides of the barrier is the difference of (51) and (53) above, and is therefore given by :

$$-\rho g\{a_1 \sin (Kx + \sigma t) - a_2 \sin (Kx - \sigma t) - a_3 \sin (\sigma t - \tfrac{1}{2}\pi - Kx)\} \times$$
$$\left\{\frac{D}{K}\, e^{-KD} - \frac{1}{K^2}(1 - e^{-KD})\right\}$$

which, by means of the relation (41*a*), reduces to

$$2ga_2 \sin (Kx - \sigma t)\left\{\frac{D}{K} \, . \, e^{-KD} - \frac{1}{K}(1 - e^{-KD})\right\} \quad . \quad . \quad . \quad . \quad . \quad (55)$$

On dividing this by the resultant horizontal force as given by (46), we find the line of action of this resultant force is at depth d where :

$$d = -\frac{De^{-KD}}{1-e^{-KD}} + \frac{1}{K} \qquad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (56)$$

Thus, though the resultant force is periodic in magnitude, it remains in the same line of action, given by (56). For large values of D the depth is approximately $\frac{1}{K} = \frac{\lambda}{2\pi}$.

The equation (55) for the resultant moment at the barrier, due to the horizontal wave-pressures on both sides, shows that this moment is purely periodic. This implies that there is no average tilt of the barrier, but only an oscillatory tilting motion. To calculate the average tilt, which is actually observed in practice, it is necessary, as in the case of the average or steady pressure, to consider waves of finite amplitude.

It is convenient at this stage to summarize the results which have so far been obtained.

(*a*) From the premises stated at the beginning of this Appendix, equations (28), (29), (35), and (36) were developed for the displacement co-ordinates of a particle on the incident- and transmitted-wave sides of an infinitely thin barrier possessing certain dynamical characteristics.

(*b*) From these equations, the expressions (40) and (41) were obtained for the pressures on the barrier on the incident- and transmitted-wave sides, and the expression (42) for the resultant horizontal force on the barrier. For small wave-heights these are all purely periodic.

(*c*) An expression for the *steady* force on the barrier, in terms of the energy dissipated there, is given in (43).

(*d*) By replacing the original barrier by that of *Fig. 11*, a relation was found between the positions of the resultant centre of wave-pressure and the mass-centre, and the radius of gyration of the breakwater. This is expressed in equation (48).

(*e*) The position of the centre of pressure on each side of the barrier was found in equations (52) and (54), in terms of the depth of the barrier and the wave-length of the incident waves. Also the position of the resultant centre of pressure for the forces on both sides of the barrier is obtained in equation (56).

(7) *The Effect of the Inertia of the Barrier.*

The relations which have so far been obtained between the amplitudes and phases of the incident, reflected, and transmitted waves have been derived from the assumptions :

(*a*) that there is no loss of energy through frictional forces at the breakwater and

(*b*) the X-displacement of the transmitted wave is equal at every instant to the X-displacement of the combined incident and reflected waves.

These two assumptions determine all the variables involved except one ; for example, they enable us to express the amplitude ratios $\frac{a_2}{a_1}$ and $\frac{a_3}{a_1}$ and the phase differences of the reflected and transmitted waves all in terms of Kx. One further relation is required to determine the variables completely. This relation must depend on the inertia of the barrier, which we shall now consider.

It is first noted that a travelling wave would be transmitted undiminished by a " mass-less " barrier of rigid form and of any width, the only effect being a change in phase depending on the width, the transmitted wave beginning, in the displaced position, exactly where the incident wave leaves off. For under these conditions the resultant force on the barrier would be zero. That is, for $M = 0$, $a_3 = a_1$ and $a_2 = 0$.

When the barrier has mass, there must be a net difference in the wave-pressures on the two sides in order to move it, and the incident wave can therefore no longer be transmitted unimpaired.

When the mass is infinite it will remain immovable and therefore there will be no transmitted wave. That is, as M approaches ∞, a_2 approaches a_1, and a_3 approaches 0.

In the application to actual breakwaters, the mass M which is really effective is not merely the mass of the actual barrier itself, but often considerably larger. This is because there is a considerable movement of water in the immediate vicinity of a real breakwater which cannot be included as part of the actual wave-motion, but may be regarded as part of the mechanism of the breakwater itself. This movement of the surrounding water is, however, produced and maintained by the forces introduced by the incident waves, and its effect can therefore be represented by adding a virtual inertia M' to the breakwater system. This effect is of course distinct from that due to any turbulent motion of the water, which would result in a loss of energy, and would have to be represented by suitable frictional forces. The value of M' will of course depend on the dimensions and shape of the cross-section of the breakwater, as well as the height and length of the incident waves. It will, for example, be large in the case of the "Bombardon" unit of cruciform cross-section, because of the large amount of water, trapped between the arms, which must move bodily with the breakwater.

To determine the total effective mass M, consider the horizontal forces acting on the whole system which is enclosed between two vertical planes A and B, one on each side of the breakwater and sufficiently far apart to include the movement of water referred to above. These planes are to be chosen so that the horizontal motion at A is identical with the horizontal motion at B. The whole system between A and B may then be regarded as a breakwater of the type already considered. If, now, this system is considered to be enclosed by two flexible membranes at A and B which have the same horizontal motions as the water particles at these places, then it is possible to consider the relation between the motion of the system as a whole and the external forces on it, without any reference at all to the relative motions taking place inside the membranes.

Now consider the motion of a thin layer of this system of depth δy. The external forces acting on it are the pressures over the two ends at A and B, which from (39) are given by:

$$p_1 dy = -\rho g\{y - a_1 e^{Ky} \sin(Kx_0 + \sigma t) + a_2 e^{Ky} \sin(Kx_0 - \sigma t)\}dy$$

and

$$-p_2 dy = \rho g\left\{y - a_3 e^{Ky} \sin\left(\sigma t - \frac{\pi}{2} - Kx\right)\right\}dy.$$

Hence the resultant horizontal force on this layer in the direction of x increasing is:

$$-\rho g\left\{a_1 e^{Ky} \sin(Kx_0 + \sigma t) - a_2 e^{Ky} \sin(Kx_0 - \sigma t) - a_3 e^{Ky} \sin\left(\sigma t - \frac{\pi}{2} - Kx_0\right)\right\}dy,$$

which, by (41a), reduces to:

$$2\rho g a_2 e^{Ky} \sin(Kx_0 - \sigma t)dy.$$

Now the horizontal motion at both A and B is given by:

$$-\sigma^2 X = a_3 e^{Ky} \cos\left(\sigma t - \frac{\pi}{2} - Kx\right),$$

and the horizontal acceleration of the layer is therefore:

$$-\sigma^2 X = -\sigma^2 a_3 e^{Ky} \sin(\sigma t - Kx_0).$$

Hence if dM is the *effective* mass of this layer, then

$$-\sigma^2 a_3 e^{Ky} \sin(\sigma t - Kx_0)dM = 2\rho g a_2 e^{Ky} \sin(Kx - \sigma t)dy;$$

that is:

$$dM = \frac{2\rho g a_2}{\sigma^2 a_3} . dy.$$

Hence, by integration the *total effective mass* for a breakwater of depth D is:

$$M = \frac{2\rho g D}{\sigma^2} . \frac{a_2}{a_3} = \frac{\lambda \rho D}{\pi} . \frac{a_2}{a_3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

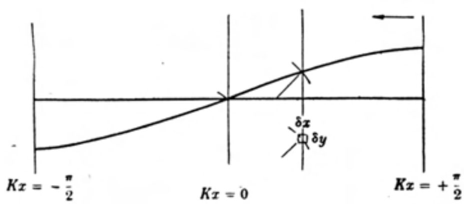
on inserting the value of σ^2.

It will be noted that this formula agrees with the limiting cases, $M = 0$ and $M = \infty$, already considered.

It is important to notice that the expression (57) for the *effective* mass M has no relation to the total actual mass of water and barrier between A and B. In fact A or B may be moved outwards by one or more whole wave-lengths without in any way affecting the result (57). This is because M is determined by the rate of change of the horizontal *momentum* between A and B, and the momentum of one complete wave is (to the first order) zero.

To understand this more clearly it is worth examining in more detail some features of the wave-motion of the water. Consider the element $\delta x \,.\, \delta y$ in the travelling wave shown in *Fig. 12.*

Fig. 12.



ELEMENT OF A TRAVELLING WAVE.

The displacement co-ordinates of this element are

$$X = ae^{Ky} \cos (Kx + \sigma t), \qquad Y = ae^{Ky} \sin (Kx + \sigma t),$$

which shows that it revolves in a circular orbit of radius ae^{Ky}. The velocity and acceleration components are respectively

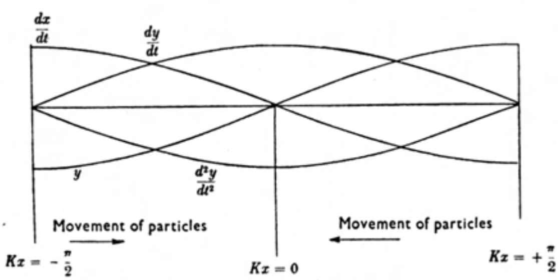
$$\dot{X} = -\sigma Y \qquad \dot{Y} = \sigma X$$

and

$$\ddot{X} = -\sigma^2 X \qquad \ddot{Y} = -\sigma^2 Y.$$

The resultant force on the element (balancing the "centrifugal force") must therefore be of magnitude $\rho \delta x \,.\, \delta y \,.\, \sigma^2 \,.\, ae^{Ky}$, and directed towards the centre of the circle. This force is the resultant of the downward gravitational force, $\rho g \delta z \delta y$, and the difference in pressures over the faces of the element as shown in *Fig. 10*, and given by equations (37) and (38).

Fig. 13.



VELOCITY AND ACCELERATION RELATIONS.

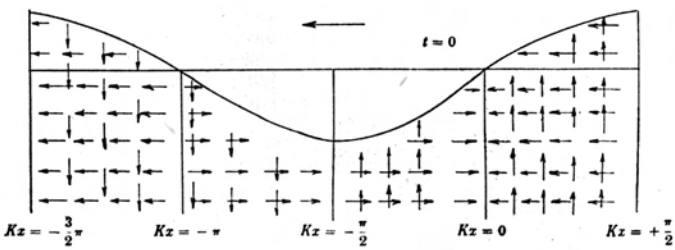
The relation between the various quantities considered above are shown diagrammatically in *Fig. 13.*

At the point $Kx = 0$ (at time $t = 0$), the particles are moving vertically upwards, and, across the vertical plane through this point, the horizontal acceleration and the pressure gradient $\frac{dp}{dx}$ have their maximum values.

At the point $Kx = \frac{\pi}{2}$, the particles are moving horizontally, but the horizontal acceleration and pressure gradient are both zero.

Fig. 14 illustrates the motion of the water in one wave-length of a travelling wave.

Fig. 14.



Particle Motions in a Travelling Wave.

The whole of the water between $Kx = -\pi$ and $Kx = 0$ (at time $t = 0$) is moving in the positive-X direction; that is, in the opposite direction from the travelling wave. The X-component of the momentum of an element $\delta x \,.\, \delta y$ at (x, y) is:

$$\rho\delta x \,.\, \delta y \,.\, \dot{X} = -\rho\delta x \,.\, \delta y \,.\, \sigma a e^{Ky} \sin(Kx + \sigma t)$$

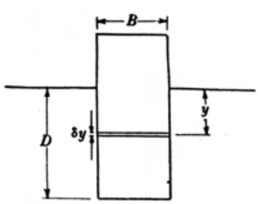
and by integrating this over one complete wave-length, it is obvious that the momentum of any layer of this length is zero. It follows that the momentum of one complete wave is zero, to the first order, as stated above.

The fact that the formula (57) for M is unaffected by shifting A or B outwards by any number of complete wave-lengths also follows from the fact that it is equal to the ratio of the difference of pressure over the ends to the acceleration, and these are the same for points distant one or more complete wave-lengths apart.

(8) *The Effect of Buoyancy.*

In *Fig. 15* is shown, in section, a simple form of floating barrier, having beam B and draught D when afloat in still water.

Fig. 15.



Simplified Buoyant Section

When subjected to wave action, such a barrier will experience the forces along the X-axis considered above but, in addition, it will be subjected to certain forces along the Y-axis. When floating undisturbed in still water the total upward force is obviously $\rho g B d$ per unit length. This upward force is equal and opposite to the barrier weight.

If now the barrier is displaced vertically from its equilibrium position it will tend to oscillate about that position, that is to say, it will have a scending motion. The usual elementary theory gives $2\pi \,.\, \frac{D}{g}$ for the period of this motion, but this ignores the motion of the water itself, and the actual period is greater than this. In any scending or rolling motion of a floating body, a corresponding wave action of the water is produced which radiates energy away from the body and thus rapidly damps the motion. The motion can be approximately represented as a rapidly damped simple-harmonic motion.

If the floating barrier is subjected to wave action, it will tend to roll and scend with the same frequency as that of the attacking waves. Thus in the case of rolling, the angular displacement will satisfy approximately a differential equation of the form

$$I\ddot{\theta} + V\dot{\theta} + S\theta = T \sin \sigma t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

where I denotes the effective moment of inertia,
V „ the damping coefficient,
S „ the stabilizing force per unit angle of heel,
and $T \sin \sigma t$ „ the moment of the periodic external force produced by wave action or other means.

The solution of this equation gives for the maximum amplitude of roll :

$$\theta_m = \frac{T}{\sqrt{\left\{V\sigma^2 + (S - I\sigma^2\right\}}} = \frac{T}{2\pi I\sqrt{\left\{\frac{V^2 f_a{}^2}{I^2} + (f^2 - f_a{}^2)^2\right\}}} \quad . \quad . \quad . \quad (59)$$

where f_a denotes the natural frequency of roll $= \frac{1}{2\pi}\sqrt{\frac{S}{I}}$

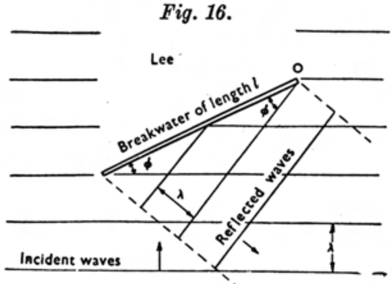
and f „ the applied frequency.

On comparison with *Fig. 11* it will be seen that the value of θ_m obtained above must not exceed $\frac{2\pi a}{\lambda} e^{-\frac{1}{2}KD}$, if the barrier is to reduce the amplitude of the transmitted wave to a or less. If this condition is to be met for all wave-lengths up to the predetermined maximum length against which the barrier is to operate, then the frequency of the longest wave must be greater than the natural frequency of roll of the barrier.

In the same way the tendency to scend of the barrier may be reduced to a minimum by making the natural frequency for scending motion smaller than the frequency of the longest incident wave it is desired to stop, and it is obvious that the wave-reducing capabilities of the barrier will be improved by reducing the scending in this way. The same requirement also applies to the natural frequency of pitch of the barrier.

If the frequencies of scend, pitch, and roll are correctly proportioned to the frequency of the incident waves, the amplitude of the transmitted wave, both immediately behind, and at some distance behind, the barrier may be determined from equation (47) using in that expression the effective instead of the actual mass of the barrier.

Fig. 16.



PLAN VIEW OF INCLINED UNIT.

(9) *Waves Incident Obliquely on the Barrier.*

The whole of the above treatment has been based on the assumption that the barrier is placed parallel to the wave-front and that it is of infinite length. The effects of placing a finite barrier at an angle to the wave front are difficult to calculate exactly, but it is, however, possible to see what some of the effects are, without going into elaborate details. Suppose a barrier of length l is placed at an angle ϕ to the wave front of the incident waves of length λ, as in *Fig. 16*.

The reflected waves will make the same angle ϕ with the barrier as the incident waves and they will extend over a band of width $l \cos \phi$, as indicated in the figure. Also the sheltered area will be of width $l \cos \phi$, but the wave reduction within this region will be affected not only by the wave motion which is transmitted through and under the barrier, but also by the waves which are diffracted round the edges of the barrier.

At any particular instant, the forces on the barrier due to the wave pressures will no longer be uniform along the length of the barrier (as was the case when $\phi = 0$), but will have a periodic variation along this length. Consider, for example, the pressures exerted by the incident waves. If the surface elevation of these waves is given by:

$$Y_S = a \sin(\sigma t + Kx),$$

and the origin O for x is taken at one end of the breakwater, as in *Fig. 16*, then the value of x at a point on the breakwater at distance r from O is:

$$x = r \sin \phi.$$

Hence the pressure on the breakwater at this point (r) and at depth $-y$, due to the incident waves, is

$$p = \rho g\{-y + ae^{Ky} \sin(\sigma t + Kr \sin \phi)\} \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Since this is a periodic function of r as well as of t, it follows that there will be a periodic turning moment tending to produce oscillations of the barrier about a vertical axis through its mid-point.

The resultant force normal to the breakwater, arising from the incident waves and tending to move the breakwater backwards and forwards bodily, is given by

$$P_1 = \int_{-D}^{Y_S}\int_{o}^{l} pdr \, . \, dy$$

$$= \rho g \int_{-D}^{Y_S} \left\{ -yl + \frac{a}{K \sin \phi} e^{Ky} \left[-\cos(\sigma t + Kr \sin \phi) \right]_o^l \right\} dy$$

$$= \rho g \int_{-D}^{Y_S} \left\{ -yl + \frac{2a}{K \sin \phi} e^{Ky} \sin(\tfrac{1}{2}Kl \sin \phi) \sin\left(\sigma t + \tfrac{1}{2}Kl \sin \phi\right)\right\} dy.$$

When $\phi \rightarrow 0$, the second term in the integrand tends to:

$$ale^{Ky} \sin \sigma t.$$

Hence, turning the breakwater through an angle ϕ to the wave front has the effect of changing the magnitude of the periodic force on the breakwater, due to the incident waves, in the ratio:

$$\frac{2 \sin(\frac{1}{2}Kl \sin \phi)}{Kl \sin \phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

This ratio is always numerically less than unity; hence the periodic force on the breakwater is reduced, and the corresponding motion of the breakwater, and the resultant transmitted wave, will likewise be reduced.

(10) *Breakwater with Gaps.*

One further point must now be mentioned. The length of any single floating barrier is limited by reasons of mechanical strength. Obviously the greatest efficiency is obtained from the greatest possible length, but there is a point where the increased efficiency is more than outweighed by the increased difficulties of construction. Hence in practice it is necessary to construct the breakwater from a number of separate floating units, with gaps between them. These gaps are themselves limited, as to their minimum dimension, by certain mechanical factors which are outside the scope of this mathematical discussion. A certain proportion of the incident wave energy will pass through these gaps. If the proportion of the total length of the breakwater which is blocked by floating barriers is R, then the proportion of total wave energy passing through the gaps is roughly $1 - R$. A diffraction pattern of waves will be formed immediately behind the breakwater, which will ultimately rebuild itself into transmitted waves of amplitude-ratio slightly less than $\sqrt{(1 - R)}$. In calculating the total amplitude of the transmitted wave, the effect of this energy passing through the gaps must be added to the other sources of energy transmission.